Name: ____ Period: ____________

AP Physics C: Mechanics

SUMMER ASSIGNMENT 2020:
Basic Physics Review + Intro to Physics Calculus

Instructions:

1. Read pages 1 – 12 (there are a few practice questions mixed in)
2. Complete the practice at the end of this packet (pages 13 – 19)

- This will be due shortly after school starts in the fall (this will *NOT* be due on the first day).
- We will be covering this material very quickly in the first couple of days in class.
- In order to be best prepared, please complete the two items outlined above.

Measurement in Physics

SI Units

- Stands for "System International"
- Three fundamental units for length, mass and time:

SI Quantity	SI Unit
Length	Meter
Mass	Kilogram
Time	Second

- There will be other units to consider throughout the course – from the Physics C formula chart:

UNIT SYMBOLS								
	meter,	m	mole,	mol	watt,	W	farad,	F
	kilogram,	kg	hertz,	Hz	coulomb,	C	tesla,	T
	second,	s	newton,	N	volt,	V	degree Celsius,	°C
	ampere,	A	pascal,	Pa	ohm,	Ω	electron volt,	eV
	kelvin,	K	joule,	J	henry,	H		

Common Prefixes

From the Physics C formula chart:

PREFIXES		
Factor	Prefix	Symbol
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

Dimensional Analysis:

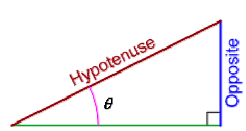
Suppose we want to convert 65 mph to ft/s or m/s.

$$65\frac{\cancel{miles}}{\cancel{hour}}\cdot\frac{1\cancel{hour}}{60\cancel{min}}\cdot\frac{1\cancel{min}}{60sec}\cdot\frac{5280\,ft}{1\cancel{mile}} \Rightarrow \frac{65\times1\times1\times5280}{1\times60\times60\times1} = 95\frac{ft}{s}$$

$$95\frac{\cancel{ft}}{s}\cdot\frac{1meter}{3.281\cancel{ft}} \Rightarrow \frac{95\times1}{1\times3.281} = 29\ m/s$$

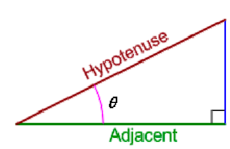
Trig Functions

Sine



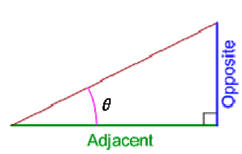
$$\text{Sin}\,\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

Cosine



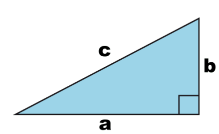
$$\text{Cos}\,\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

Tangent



$$\text{Tan}\,\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Pythagorean Theorem



$a^2 + b^2 = c^2$

From your formula chart	

VALUES OF TRIGONOMETRIC FUNCTIONS FOR COMMON ANGLES

θ	0°	30°	37°	45°	53°	60°	90°
$\sin\theta$	0	1/2	3/5	$\sqrt{2}/2$	4/5	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	4/5	$\sqrt{2}/2$	3/5	1/2	0
$\tan\theta$	0	$\sqrt{3}/3$	3/4	1	4/3	$\sqrt{3}$	∞

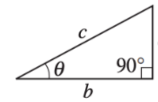
Right Triangle

$a^2 + b^2 = c^2$

$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$

$\tan\theta = \frac{a}{b}$



Scalars & Vectors

- **<u>Scalar</u>**: any quantity in physics that has magnitude only (no direction)
- Magnitude = a numerical value with units

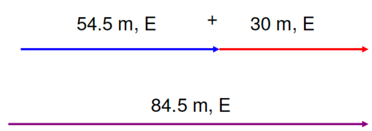
Scalar Example	Magnitude
Speed	20 m/s
Distance	10 m
Age	15 years
Heat	1000 calories

- **<u>Vector</u>**: any quantity in physics that has both **magnitude and direction**
- Typically indicated by drawing an arrow above the symbol: $\vec{v}, \vec{x}, \vec{a}, \vec{F}$

Vector	Magnitude & Direction
Velocity	20 m/s, N
Acceleration	10 m/s/s, E
Force	5 N, West

Vector Addition: If two similar vectors point in the **SAME direction**, **add** them.

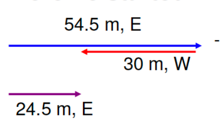
Example: A man walks 54.5 meters east, then another 30 meters east. Calculate his displacement relative to where he started?



Notice that the SIZE of the arrow conveys MAGNITUDE and the way it was drawn conveys DIRECTION.

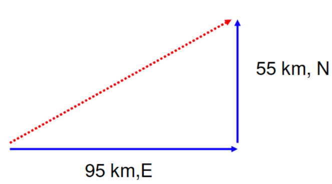
Vector Subtraction: If two similar vectors point in **OPPOSITE directions**, **subtract** them.

Example: A man walks 54.5 meters east, then 30 meters west. Calculate his displacement relative to where he started?



When two vectors are **perpendicular**, you must use the **PYTHAGOREAN THEOREM**.

A man walks 95 km, East then 55 km, north. Calculate his RESULTANT DISPLACEMENT.

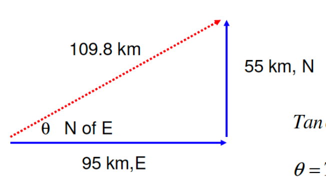


$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

$$c = \text{Resultant} = \sqrt{95^2 + 55^2}$$

$$c = \sqrt{12050} = 109.8\ km$$

Just putting North of East on the answer is NOT specific enough for the direction. We MUST find the VALUE of the angle.



To find the value of the angle we use a Trig function called TANGENT.

$$Tan\,\theta = \frac{opposite\ side}{adjacent\ side} = \frac{55}{95} = 0.5789$$

$$\theta = Tan^{-1}(0.5789) = 30^\circ$$

$$109.8 km @ 30^\circ\ NofE$$

1. Greek Alphabet

Memorize it, practice writing it, practice saying it, and learn to love it. We won't use all of the symbols this year, but it will help you to keep up with new content and equations…plus it will help you in college if you join a fraternity or sorority.

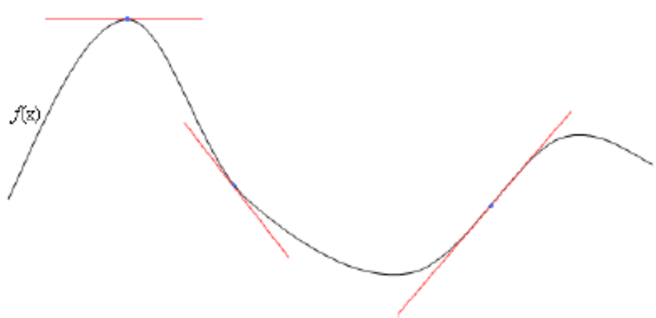
A	ALPHA	al-fah	N	NU	new
B	BETA	bay-tah	Ξ	XI	zzEYE
Γ	GAMMA	gam-ah	O	OMICRON	omm-i-cron
Δ	DELTA	del-ta	Π	PI	pie
E	EPSILON	ep-si-lon	P	RHO	roe
Z	ZETA	zay-tah	Σ	SIGMA	sig-mah
H	ETA	ay-tah	T	TAU	taw
Θ	THETA	thay-tah	Y	UPSILON	opp-si-lon
I	IOTA	ee-o-tah	Φ	PHI	fie
K	KAPPA	cap-ah	X	CHI	kEYE
Λ	LAMBDA	lamb-dah	Ψ	PSI	sigh
M	MU	mew	Ω	OMEGA	o-may-gah

alpha	A	α	nu	N	ν
beta	B	β	xi	Ξ	ξ
gamma	Γ	γ	omicron	O	o
delta	Δ	δ	pi	Π	π
epsilon	E	ε	rho	P	ρ
zeta	Z	ζ	sigma	Σ	σ
eta	H	η	tau	T	τ
theta	Θ	θ	upsilon	Y	υ
iota	I	ι	phi	Φ	ϕ
kappa	K	κ	chi	X	χ
lambda	Λ	λ	psi	Ψ	ψ
mu	M	μ	omega	Ω	ω

2. Rules of Differentiation

In algebra, you were taught how to find the slope of a straight line, both by interpretation of a linear function (for example, we know that $y = 2x + 3$ has a slope of 2 because the slope is the coefficient of the x term) and with the slope formula, $m = (y_2 - y_1)/(x_2 - x_1)$. However, for non-linear functions, the first method cannot be used, and the second can only provide an average slope over an interval.

Calculus provides us with a way to find the slope at any point along any function, whether it is linear or not. This is done by measuring the slope of a line tangent to the curve at that point. Examples of a few tangent lines are shown below.

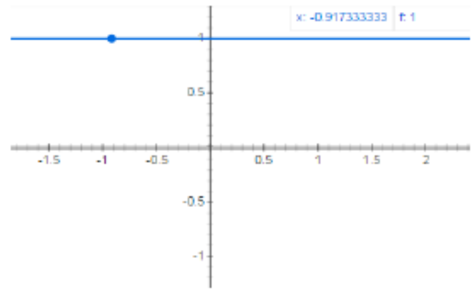


Each of the lines is tangent to the graph of $f(x)$ at the corresponding point. Notice that the lines seem to follow where the function would be if it continued in either direction with a constant slope. In calculus, we use a method known as differentiation to allow us to find the slopes of these tangent lines.

Before we begin learning how to differentiate, it is important to understand a few things. Looking at a linear function, slope is a rate of change. More specifically, it is the rate of change of y with respect to x. In simpler terms, the slope of a line tells us how much the y value changes for each increment that x changes. For linear functions, this value remains constant. However, the rate of change of a non-linear function can't be a constant, leaving only one possibility: it is another function.

This brings us to differentiation, which is the process of using a given function to find the function representing its rate of change, called its derivative. For a function *f*(x), the derivative of *f*(x) is denoted *f′(x)*, pronounced "f prime of x". Here, we will only cover simple polynomials, as they are the primary type of function on the AP Physics C exam. Later during the year we will review how to take derivatives of trigonometric functions.

Rule #1: The Constant Rule
If *f*(x) = c, where c is any constant, *f* ′(x) = 0



Example: *f*(x) = 1

The graph to the right shows the function f(x) = 1. The graph is a horizontal line, so the slope of the line at all points is equal to zero. Therefore, the derivative *f* ′(x) = 0

Example: *f*(x) = -10 → *f* ′(x) = 0

Example: *f*(x) = 3p, where p is a constant → *f* ′(x) = 0 *(any number multiplied by a constant is also a constant)*

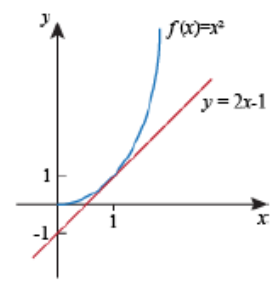
Rule #2: The Power Rule
If *f*(x) = x^n, then *f* ′(x) = nx^{n-1}

So if *f*(x) = x^3, then using the power rule, *n* = 3 and *n*-1 = 2, which means the derivative *f* ′(x) = $nx^{n-1} = 3x^{3-1} = 3x^2$

Example: *f*(x) = x^2, so *f* ′(x) = $2x^1 = 2x$ *(no exponent written on x → x^1)*

If we wanted to apply this equation at a particular point, we can find the slope of the tangent line at that point.

For example, when x = 1, *f* ′(1) = 2(1) = 2 so the slope of the tangent line is 2. You can then substitute the slope and the (x,y) coordinates for this point into the y=mx+b formula for a line to find the y-intercept.



When x = 1, *f(x)* = 1 so b = y – mx = 1 – (2)(1) = -1

The equation for the tangent line at x=1 is y = 2x-1, which has been plotted on the graph, as shown.

If this were a physics problem, then x could be time and y could be position, which means the slope would be velocity. The slope at time = 1 second would be the rate of change of position over time, aka velocity. Therefore the derivate would give us the slope of the tangent line, which represents the instantaneous velocity, in this case 2 m/s.

If we were to evaluate the derivative at time = 2 seconds, then we would find that the derivative *f ′(2)* = 2(2) = 4, which in our application would mean that at t = 2 seconds, the instantaneous velocity would now be 4 m/s.

This is a huge improvement over algebra-based physics, in which we could find only the instantaneous velocity if velocity was constant (straight line portion of a position vs. time graph) or average velocity if the graph of position vs. time was curved.

Example: *f*(x) = x, so *f* (x) = $1x^0$ = 1 *(any number to the power of 0 is 1, so x^0 = 1)*

Rule #3: The Constant Multiple Rule
If *f*(x) = c·*g*(x), where c is some constant, then *f* ′(x) = c·*g*′(x)

Example: *f*(x) = $2x^2$, so *f* ′(x) = $2(2x^1)$ = 4x

Example: *f*(x) = 5x, so *f* ′(x) = $5(1x^0)$ = 5

Rule #4: The Sum and Difference Rules
If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$
If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$

This rule looks more complicated than it actually is because it essentially says that you can treat each term in a polynomial as individual functions added together, allowing you to apply the power rule to each term.

Example: $f(x) = x^3 + x^2$

Use the power rule on $x^3 \rightarrow 3x^2$ and $x^2 \rightarrow 2x^1$
Add the result of the two power rules such that $f'(x) = 3x^2 + 2x$

Example: $f(x) = 2x^2 - 3x$, so $f'(x) = 4x - 3$

Example: $f(x) = 5x^4 + 2x^3 - x^2 - 3x + 5$, so $f'(x) = 20x^3 + 6x^2 - 2x - 3$ *(the derivative of 5 is 0 since it is a constant)*

Using just these four simple rules, it is possible to take the derivative of any simple polynomial containing only one variable, like those in the examples for rule #4. More complex polynomials can be converted into simpler polynomials with a little manipulation.

If the function has parentheses, it can be turned into a simple polynomial by distributing.

Example: $f(x) = (6x)(x+3) = 6x^2+18x$, such that $f'(x) = 12x+18$.

If the function has a root, it can be turned into a simple polynomial by converting to a power.

Example: $f(x) = -2\sqrt[4]{x} = -2x^{1/4}$, such that $f'(x) = -2(1/4)x^{1/4-1} = -1/2\, x^{-3/4}$.

Determine the derivative for each of the following functions and write the result in the blank provided in simplified format. <u>Answers should be in the form of an equation</u>, not an expression (For example, $f'(x)=cx^2$ rather than just cx^2).

a. $f(x) = 5$ __________________________

b. $v(t) = 8t + 3$ __________________________

c. $y = 5/4\, x^2$ __________________________

d. $p(v) = 65v$ __________________________

3. Rates of Change

The derivative represents a rate of change of a dependent variable with respect to changes in an independent variable. For example, the flow rate of water through a pipe would be the rate of change of the volume with respect to time if the volume of water exiting the pipe is a function of time.

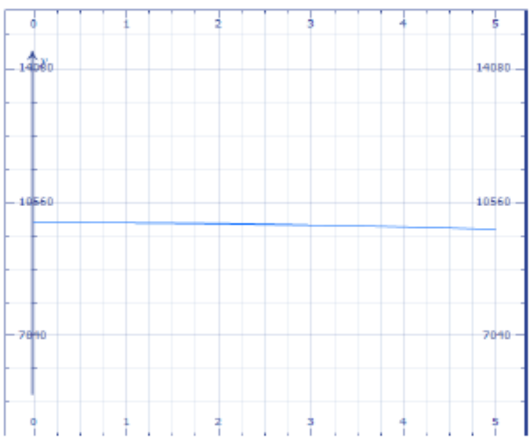
Example: Water is flowing out of a water tower in such a way that after t minutes the volume of water in the tower in gallons is modeled by the equation $V(t) = 10{,}000 - 10t - t^3$. Determine how fast the water is flowing after 2 minutes.

The speed the water is flowing would be calculated as dV/dt, so we need to take the derivative of the volume equation with respect to time.

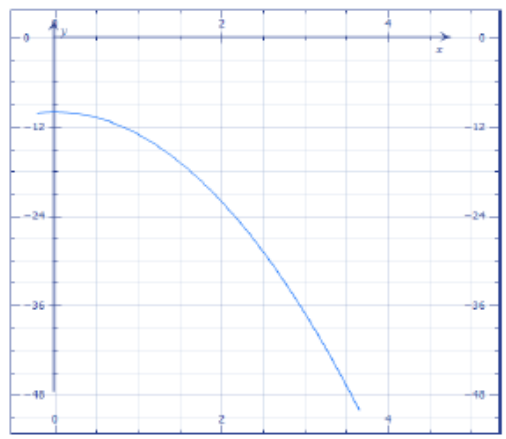
$$flow\ rate = \frac{dV}{dt} = \frac{d}{dt}(10{,}000 - 10t - t^3) = -10 - 3t^2$$

The result above gives the flow rate out of the water tower in units of gallons/minute for any time t. The question wants the flow rate at 2 minutes, so we just need to substitute in $t = 2$ minutes into the equation above, and we find that the flow rate at this time is -22 gal/min.

But what about the flow rate at other times? Sometimes graphing a function that is provided or derived can give meaningful information to answer conceptual questions, and you are often required to choose appropriate graphs or sketch graphs for different scenarios on the AP exam.



Graph of volume vs. time with volume in gallons plotted on the y-axis and time in minutes plotted on the x-axis.



Graph of flow rate vs. time with flow rate in gal/min plotted on the y-axis and time in minutes plotted on the x-axis.

The graphs above represent the volume and flow rate equations for the water tower example. Since the volume equation is non-linear, its slope, represented by the flow rate, is constantly changing. As time increases, the flow rate is increasingly negative as a larger volume of water leaves the water tower as each minute passes. Substituting in $t = 3$ minutes into the flow rate equation gives a flow rate of -37 gal/min, which is larger than at 2 minutes.

While you will not be calculating flow rates in AP Physics C, this process of applying derivatives and analyzing their meaning is a valuable skill. Let's take a look at some physics examples from kinematics and other units and practice applying derivatives. You may wish to review kinematics and an AP Physics C formula sheet prior to starting this section.

The most basic way to apply derivatives to physics is through kinematics when analyzing position, displacement, velocity, and acceleration. Upon analyzing these properties of motion, you may notice that velocity is the rate of change of position over time, and acceleration is the rate of change of velocity over time. This means that velocity is the derivative of position with respect to time, and acceleration is the derivative of velocity with respect to time.

Knowing this information, we see that for any position function $x(t)$, $x'(t) = v(t)$, the velocity function, and $v'(t) = a(t)$, the acceleration function.

For example, one of our favorite kinematic equations relates position and time: $x = x_0 + v_0t + \frac{1}{2}at^2$. This equation is the same on the AP Physics 1 formula sheet as on the AP Physics C formula sheet. Note that for any problem using this equation, a, v_0 and x_0 are all constants, only x and t are variables. We can rewrite this function in more of a math format and take the derivative:

$$x(t) = \tfrac{1}{2}at^2 + v_0t + x_0$$
$$x'(t) = \tfrac{1}{2}a(2t^1) + v_0(1t^0) + 0$$
$$= at + v_0$$

$$\boxed{\begin{aligned} &\upsilon = \upsilon_0 + at \\ &x = x_0 + \upsilon_0 t + \frac{1}{2}at^2 \\ &\upsilon^2 = \upsilon_0^{\ 2} + 2a(x - x_0) \end{aligned}}$$

Since $v(t) = x'(t)$, this gives us $v(t) = v_0 + at$, which is another kinematic equation on the formula chart.

For any graph, the derivative of the function would give you the slope of the tangent line, which represents the instantaneous rate of change of the y-axis variable vs the x-axis variable. So the derivative can be applied throughout physics and is not limited to kinematics.

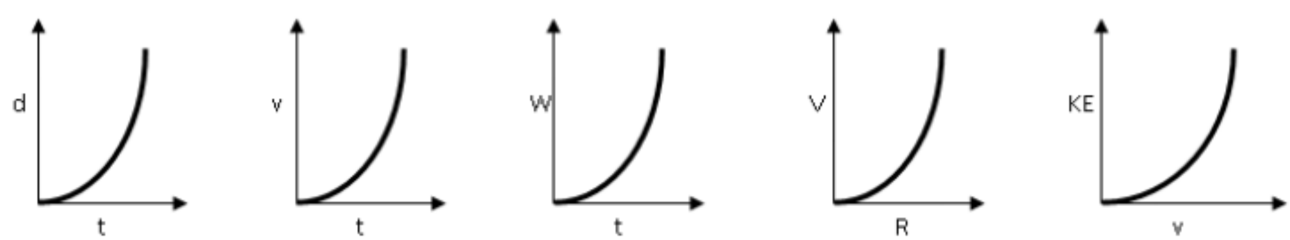
Without Calculus: $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ is the average rate of change of y with respect to x over the interval (x_1, x_2)

With Calculus: $\dfrac{dy}{dx}$ is the rate of change of y with respect to x; also the instantaneous rate of change.

Note on notation: The derivative x (dependent variable) with respect to t (independent variable) can be written in several ways: *x'(t), x' or dx/dt*. I generally prefer the first or third methods of notation because they indicate both the dependent and independent variables.

There are more physics variables that are defined as rates of change. For example, current is the rate of flow of charge over time ($I = dq/dt$), power is the rate of change of energy over time ($P = dU/dt$), and the strength of the electric field is negative the rate of change of electric potential with respect to distance ($E = -dV/dx$). Some of these calculus-based definitions of variables are present on the AP formula sheet, others you will be expected to memorize.

For each of the graphs below, identify the variable that is represented by the derivative of the dependent variable with respect to the independent variable.



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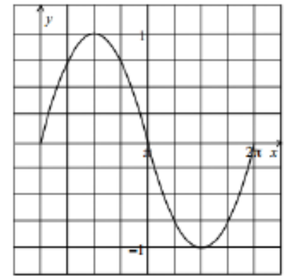
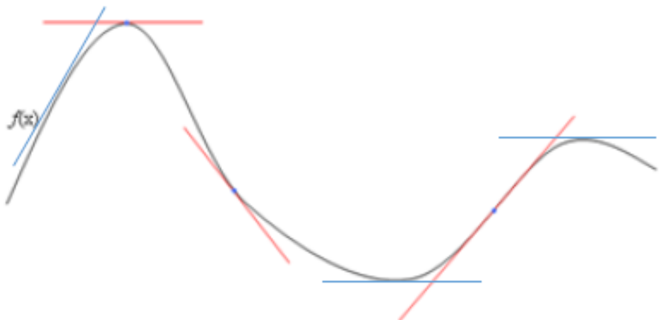
4. Minima and Maxima

Often a function reaches a local maximum or a local minimum. The sine function in the graph to the right has a maximum at $x = \pi/2$ and a minimum at $x = 3\pi/2$. Extrema are easily recognized on a graph, but you can identify maximums and minimums analytically without graphing using derivatives.



function: $y = \sin x$

Recall that the derivative of a function is the slope of the tangent line at a particular point. Several tangent lines are show on the diagram below.



The tangent line drawn for any maximum or minimum will have a slope of zero.

The tangent lines on either side of a maximum or minimum will switch signs from + to – for a maximum, and from – to + for a minimum, as you move from left to right on the graph.

Since the slopes of the tangent lines are zero at a local maximum or minimum, this means that the derivative is also zero at these points.

You will learn a very systemic approach to evaluating maxima and minima using the First and Second Derivative Tests in your calculus course. The following abbreviated procedure works for smooth line, continuous curves with no sharp points or discontinuities.

To find a maximum or minimum of a function $y(x)$:
1. Take the derivative $y'(x)$
2. Set the derivative equal to zero.
3. Solve for x.
   → The x values found using these methods are called critical points
4. Substitute the x values into the original function $y(x)$ to find the corresponding y values.
   → These values correspond to the critical points, but not all critical points are maxes or mins.

In some cases, it may be necessary to determine whether the y values corresponding to the critical points are maxes, mins, or neither. This is needed for more complex functions for which there may be multiple max/min points for the range of values you are interested in. You will most likely be working with quadratic equations, which will have either one minimum or one maximum. The last step below can be used to verify whether a y-value corresponding to a critical point is a maximum or minimum.

5. Choose an x value just to the left of the critical point and another to the right of the critical point and substitute into the derivative $y'(x)$.
   → Maximum: $y'(x)$ is (+) to the left of the critical point and (-) to the right of the critical point
   → Minimum: $y'(x)$ is (-) to the left of the critical point and (+) to the right of the critical point
   → Neither: $y'(x)$ has the same sign on either side

It is much easier to find maxima and minima using the graphing functions of your favorite graphing calculator, and this works for functions with numerical values, such as $f(x) = 6x^2 + 2$, but would be trickier (although still possible with some tweaks and substitutions) for non-numerical equations, such as $v(t) = -kt^2 + b$. When you work the practice problems after the examples, you can (and should) check your answers using your calculator, but you need to work the problems by hand.

5. Indefinite Integrals (Antiderivatives)

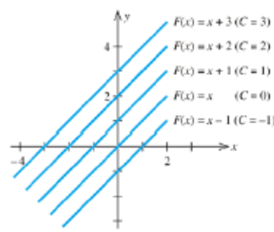
Differentiation is a useful tool for physics, however it cannot solve all of our problems. For example, what if you want to go backwards, such as from a velocity function to its position function? Differentiation cannot go backwards like this, however, calculus has another tool for this task: integration. There are two broad types of integration: indefinite and definite. Here, we will start with indefinite integration and its applications, then continue to definite integration and its applications.

Like differentiation, indefinite integration is a process that follows certain rules. Integration also has its own special notation, which looks something like the following: $\int f(x)dx$. The symbol $\int$ tells you that you need to integrate, $f(x)$ is the function to be integrated, and dx tells you that you are integrating with respect to x. As long as the variable after d matches the variable in the function and there are no other variables in the function, the following rules apply exactly.

Rule #1: Constant Rule of Integration
$\int k\, dx = kx + c$, where k and c are constants.

Example: $\int 1\, dx = 1x + c = x + c$

Note the $+c$ that appears after integration. This is called the constant of integration, and must be added at the end of the solution to any indefinite integration problem.



The following functions all have a slope of 1: $f(x) = x$, $f(x) = x + 1$, $f(x) = x + 3$, etc. Therefore when you take the derivative of any of these functions, $f\,'(x) = 1$ for all of them.

Taking the integral undoes the derivative giving you the original function back. However, you can't get the constant back because the derivative of any constant is 0. We therefore place a $+c$ after the solution to the integral to represent all the possible solutions. This is called the general solution. The image to right shows a selection of possible solutions to the integral in this example for different values of the constant. Note that they all have a slope of 1. A specific solution can be found by substituting in values for c.

A good way to check to see if you did the integration correctly is to take the derivative of your answer. If you get the original function, then your solution was correct. <u>Do not forget the $+c$!</u>

Example: $\int -7\, dx = -7x + c$

Example: $\int 4p\, dx = 4px + c$ *(Assuming that p is a constant)*

Example: $\int dx = \int 1\, dx = x + c$

Rule #2: Power Rule of Integration
$\int x^n\, dx = (x^{n+1})/(n+1) + c$

Example: $\int x^2 dx = (x^{2+1})/(2+1) + c = x^3/3 + c$

Example: $\int x^3 dx = x^4/4 + c$

Example: $\int x\, dx = \int x^1 dx = x^2/2 + c$

Note that this rule does not work for integrating $x^{-1} = 1/x$, as it would result in $x^0/0$, which is undefined. There is a separate rule for this involving natural logarithms, but is not necessary to cover at this time.

Rule #3: Zero Rule of Integration
$\int 0\, dx = c$

The derivative of any constant is zero, so the antiderivative (integral) of zero is all constants.

Rule #4: Constant Multiple Rule of Integration
$\int k\, f(x)\, dx = k \int f(x)\, dx$

Example: $\int 2x\, dx = 2\int x\, dx = 2(x^2/2) + c = x^2 + c$

Example: $\int 5x^2\, dx = 5\int x^2 dx = 5x^3/3 + c$

Rule #5: Sum and Difference Rules of Integration
$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
$\int f(x) - g(x)dx = \int f(x)dx - \int g(x)dx$

This rule is very similar to the sum and difference rule for derivatives.

Example: $\int (x^3 + x^2)\, dx$

> Use the power rule on $x^3 \rightarrow x^4/4$ and $x^2 \rightarrow x^3/3$
> Add the result of the two power rules and the constant such that $\int (x^3 + x^2)\, dx = x^4/4 + x^3/3 + c$
>
> You will only get one constant each time, not one for each term, although the final constant is the sum of the two constants from the intermediate steps.
>
> Parentheses are not always used to isolate the integrand. Instead, the polynomial to be integrated is bookmarked by the $\int$ symbol and *dx*.

Example: $\int 3x^2 + 5\, dx = 3x^3/3 + 5x + c = x^3 + 5x + c$

Example: $\int 2x^3 + 4x^2 + 3x + 1\, dx = 2x^4/4 + 4x^3/3 + 3x^2/2 + x + c = x^4/2 + 4x^3/3 + 3x^2/2 + x + c$

6. Applications of the Indefinite Integral

The indefinite integral is useful, but in physics, we don't want all possible solutions, we want the particular solution that fits the given conditions for the problem, so we need a method for finding *c*.

In order to find *c*, you need to know some additional information, which is given as a set of initial conditions. For example, the problem might be to integrate $f'(x) = 3x + 2$, and the initial condition might be $f(1) = 3$. So in order to solve this, start with your given function:

Given:	$f'(x) = 3x + 2$	
Setup the integral:	$f(x) = \int f'(x)\, dx = \int 3x + 2\, dx$	
Integrate:	$f(x) = 3x^2/2 + 2x + c$	← *This is the general solution*
Substitute initial condition:	$f(1) = 3(1)^2/2 + 2(1) + c$	
Simplify:	$3 = 3/2 + 2 + c$	
Solve for c:	$c = -1/2$	
Substitute c into f(x):	$f(x) = 3x^2/2 + 2x - ½$	← *This is the particular solution*

Now we know exactly what $f(x)$ is, with only one variable.

Next, we must apply this to physics. The most basic application is again in kinematics. Remember that the derivative of position is velocity, and the derivative of velocity is acceleration. Now, notice that integration is, essentially, the reverse of differentiation, so that means that the integral of acceleration is velocity, and the integral of velocity is position.

For example, consider an object that is dropped from rest from a height of 100 m. If we wanted to derive an equation for the velocity of the object at any time *t*, we could take the derivative of the vertical position function, if it was given, or use a kinematic equation. However, we can also start with the acceleration equation and integrate to find the velocity equation, but we don't know that either…or do we?

If we assume the object acts as a projectile, then the acceleration of the object is a constant -9.8 m/s². We can therefore define $a(t) = -9.8$ and integrate with respect to t.

Given:	$a(t) = -9.8$
Integrate:	$\int a(t)\, dt = \int -9.8\, dt = -9.8t + c$

Since the integral of acceleration is velocity, then $v(t) = -9.8t + c$. But we need to eliminate *c* from the equation so that we get only the solution that matches the initial conditions. But what are the initial conditions?

Since the object was dropped from rest, we know that the starting velocity is zero, which means that when $t = 0$, $v(0)=0$, and substituting in:

General Equation:	$v(t) = -9.8t + c$
Substitute Initial Condition:	$v(0) = -9.8\,(0) + c$
Solve:	$0 = c$

Therefore since $c = 0$, the velocity function for this particular example is $v(t) = -9.8t$ with units of m/s.

Furthermore, we can continue this example to derive a function for the vertical position of the object at any time t since the integral of the velocity function is position.

Given:	$v(t) = -9.8t$
Integrate:	$\int v(t)\, dt = \int -9.8t\, dt = -9.8(t^2/2) + c = -4.9t^2 + c$

Therefore the general position function is $y(t) = -4.9t^2 + c$, but we need to find the particular solution that meets the initial condition.

We know that the object started at a height of 100 m, which means that at $t = 0$, $y(0) = 100$, so substituting in:

General Equation:	$y(t) = -4.9t^2 + c$
Substitute Initial Condition:	$y(0) = -4.9(0)^2 + c$
Solve:	$100 = c$

Since $c = 100$, the vertical position function for this particular example is $y(t) = -4.9t^2 + 100$.

Therefore, $y(t)$, $v(t)$, and $a(t)$ can be derived from a single equation and some initial conditions, thus describing the motion at any point in time.

7. Definite Integrals

Now that we have learned indefinite integration, we can begin learning definite integration. Definite integration uses the same set of rules as indefinite integration, but the notation and method have some additions. Also, the answer to a definite integral is not a function, but a number. We will talk more about the meaning of this number later. First, take a look at the notation for a definite integral:

$$\int_{x_1}^{x_2} f(x)dx$$

Notice that the only change in notation is the addition of x_1 and x_2 at the top and bottom of the integral symbol. These are called the limits of integration, and define the interval on which you are performing the definite integration. x_1 is the lower limit, and thus is usually a lower number, and x_2 is the upper limit, and thus usually a higher number. This idea will become more clear as we practice a few problems, but in order to do this, we must look at arguably the most important theorem in calculus:

The Fundamental Theorem of Calculus

$$\int_{x1}^{x2} f'(x)dx = f(x_2) - f(x_1)$$

This theorem instructs us to take the integral, but instead of adding a *+c* onto the end of the antiderivative, you substitute the limits of integration into the antiderivative and subtract to get a single, numerical answer. The following example will be used to demonstrate why the *+c* is no longer necessary.

Example: $\int_2^7 3\, dx$

Indefinite Integration: $\int 3\, dx = 3x + c$		
Substitute in limits:	$= (3*7 + c) - (3*2 + c)$	
Distribute the (-):	$= 21 + c - 6 - c$	*(c cancels from the solution)*
Simplify:	$= 21 - 6$	
	$= 15$	

Note that the *c* from each equation cancels out because of the distribution of the negative sign to the second equation. This will happen in every problem, so the *+c* can be ignored in definite integration. Instead of writing the antiderivative and eliminating the *+c* algebraically, you can use the notation $f(x)|_{x_1}^{x_2}$ to indicate that you will evaluate the antiderivative between the two limits of integration.

Example: $\int_2^5 -3x + 4\, dx = \frac{-3x^2}{2} + 4x\Big|_2^5$ ← *Integrate, showing limits*

$= \left[\frac{-3(5)^2}{2} + 4(5)\right] - \left[\frac{-3(2)^2}{2} + 4(2)\right]$ ← *Substitute in limits*

$= -39/2$ ← *Solve*

$= 19.5$

Note that in calculus, you would normally leave the answer as a simplified improper fraction, like -39/2. In physics, we generally use decimal answers, like -19.5. Also, be very careful to distribute the negative to all terms of the second equation. Failing to distribute this negative properly can result in radically incorrect answers.

Example: $\int_2^4 \sqrt[4]{x^3} + 2x\, dx = \int_2^4 x^{3/4} + 2x\, dx$

$= \frac{x^{7/4}}{7/4} + \frac{2x^3}{3}\Big|_2^4$

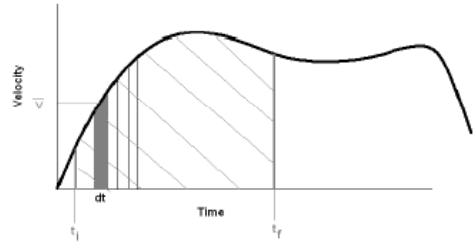
$= \left[\frac{(4)^{7/4}}{7/4} + \frac{2(4)^3}{3}\right] - \left[\frac{(2)^{7/4}}{7/4} + \frac{2(2)^3}{3}\right]$

$= 16.5$

Simplifying the above expression algebraically would be painful. Feel free to use your calculators to evaluate the function at the limits to get the decimal answer.

8. Area Under the Curve

Now that we have seen how to use definite integration on polynomials, we need to understand what it means and when it is applicable to physics. The more relevant explanation of what a definite integral means would be the total change in *f*(x) from x_1 to x_2. Graphically, the definite integral represents the area under the curve *f'(x)* between the limits of integration.



For example, if you are given a velocity function, *v*(*t*), and took the definite integral from t_i to t_f, the result would be the change in position (aka displacement) over the time interval.

If you did the same to an acceleration function, *a*(*t*), then you would be finding the change in velocity over that time interval.

In algebra-based physics, we could only find the actual area under the curve for a given graph if the graph was a standard geometric shape such as a rectangle or triangle or we approximated the area under the curve with a geometric shape. With calculus, we can find the area under the curve of any function without relying on a graph.

Example: Find the area under y=3x² between x=1 and x=5.

Setup the integral: $Area = \int_1^5 3x^2dx$

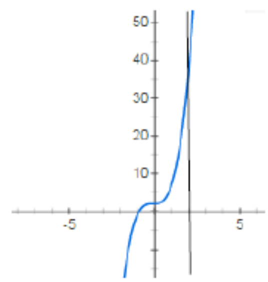
Evaluate: $= \frac{3x^3}{3}\Big|_1^5 = x^3\Big|_1^5$

Substitute: $= (5)^3 - (1)^3$

Solve: $= 124$

Example: If my velocity is given by $v = 4t^3+t^2+2$ in m/s, how far am I from the starting point after 2 seconds?

The question is asking for my displacement at t = 2 seconds, which would be the area under the curve of the velocity function. If I look at a graph of this function, the area under the curve is the area in the first quadrant bounded by the x-axis, y-axis, the curve and the line x=2. Since this is not a simple shape I need to integrate the velocity function between the limits of t = 0 seconds and t = 2 seconds to find the displacement.



$$Displacement = \int_0^2 4t^3 + t^2 + 2\ dt$$

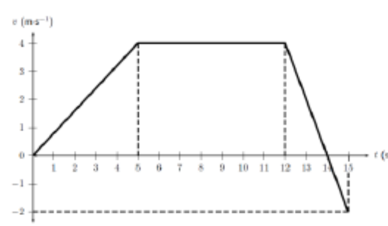
$$= \frac{4t^4}{4} + \frac{t^3}{3} + 2t\Big|_0^2$$

$$= t^4 + \frac{t^3}{3} + 2t\Big|_0^2$$

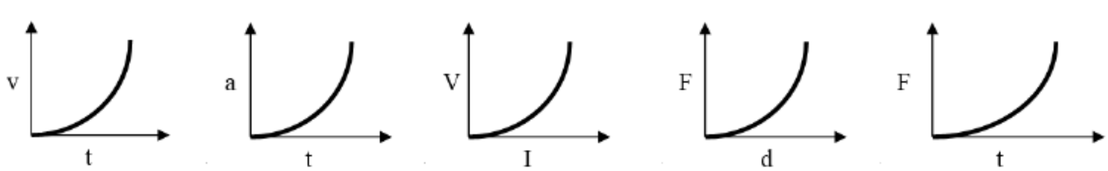
$$= \left[2^4 + \frac{2^3}{3} + 2(2)\right] - \left[0^4 + \frac{0^3}{3} + 2(0)\right]$$

$$= 22.7 \text{ m}$$

Note that the displacement is not always the same thing as the total distance travelled. In the graph to the right, the velocity function passes below the x-axis between t=14 and t=15 seconds, which would yield a negative value for the integral between those limits. In terms of displacement, this is a negative displacement indicating that the object is moving back towards the starting point, such that the displacement from t=0 to t=15 seconds is smaller than the displacement from t=0 to t=14 seconds. Therefore the distance travelled by the object after 15 seconds has passed is larger than the magnitude of the displacement.



For each of the graphs below, identify the variable that is represented by the area under the curve.



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Name: ______________________________ Period: ____________

AP Physics C: Mechanics

Introduction to Derivatives

Differentiate (find the derivative of) each of the following functions with respect to t. (HINT: Remember, $\frac{1}{x^2} = x^{-2}$ AND $x^{\frac{1}{2}} = \sqrt{x}$)

1. $x(t) = 5t^{18}$	2. $x(t) = 4t^5 + t$	3. $x(t) = 4t^5 - 5t - 3$
4. $x(t) = 3t^{\frac{5}{4}}$	5. $x(t) = t^{\frac{2}{3}}$	6. $x(t) = 4t^2 - 5t^4$
7. $x(t) = -4t^{-5}$	8. $x(t) = -2\sqrt[4]{t}$	9. $x(t) = \frac{5}{4}t^{\frac{2}{3}}$
10. $x(t) = \frac{3}{x^3}$	11. $x(t) = -3t^5 - 5t^2$	12. $x(t) = -\frac{3}{t^2} - \frac{4}{t^4}$
13. $x(t) = 4ta^{3a}$	14. $x(t) = -t^3 + 5t^2 + 8t - 48$	15. $x(t) = 7$

Name: ____ Period: __________

AP Physics C: Mechanics

INTEGRALS + DERIVATIVES + KINEMATICS REVIEW

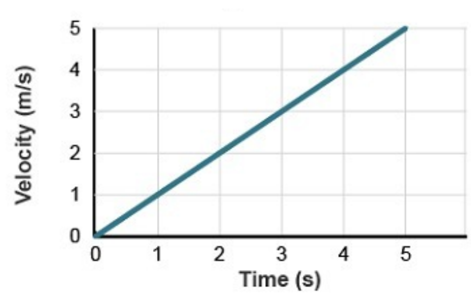
Integrate each of the following functions with respect to x.

1. $\int -6x dx$	2. $\int x^5 dx$	3. $\int (5x^8 - 2x^4 + x + 3) dx$
4. $\int -24x^5 dx$	5. $\int (x^3 + 2x) dx$	6. $\int (x^4 - x^3 + x^2) dx$
7. $\int 4x^{-5} dx$	8. $\int (3x^{-2} - 4x^{-3}) dx$	9. $\int (-9x^2 + 10x) dx$

Solve for dx/dt for each of the following functions.

10. $x(t) = 8t^6$	11. $x(t) = 4t^{-3} + 2t^{1/2}$	12. $x(t) = 2t - 5t^2$
13. $x(t) = {}^{1}/_{t^3}$	14. $x(t) = \sqrt[5]{t}$	15. $x(t) = -\frac{1}{2}t^4 + 3t^{5/3} + 2t$
16. $x(t) = 7t + 6$	17. $x(t) = \frac{1}{2}t^{-2}$	18. $x(t) = \frac{2}{3}t^3 + 5t - t^{-3}$

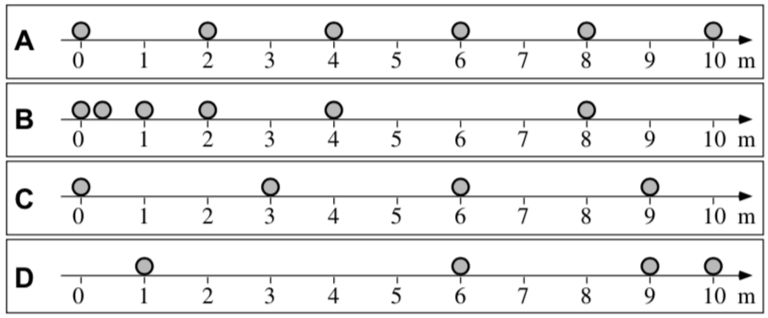
19. A car is sitting at a red light on Woodlands Parkway. When the light turns green, the car begins to accelerate at a constant rate, as illustrated by the velocity vs. time graph to the right.



a. Solve for the car's acceleration?

b. Use the graph to estimate the car's displacement from t = 0 seconds to t = 5 seconds?

c. Sketch an acceleration vs. time graph representing the motion of the car. What is the significance of the area under the curve for this graph (i.e, what variable does it represent)?



20. In each case shown above, a sphere is moving from left to right next to a tape marked in meters. A strobe (flash) photograph is taken every second, and the location of the sphere is recorded. The total time intervals shown are not the same for all spheres.
    a. Rank the magnitude of the displacement over the first 3 seconds from greatest to least. Explain your reasoning.

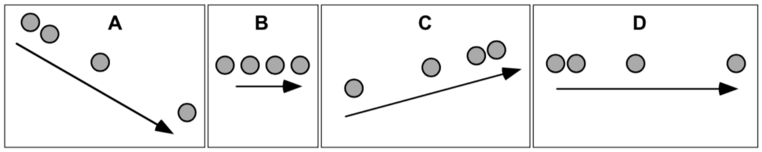
    b. Rank the magnitude of the average velocity over the first 3 seconds from greatest to least. Explain your reasoning.

    c. Rank the magnitude of the average velocity over the first 2 seconds from greatest to least. Explain your reasoning.

21. The first 10 meters of a 100-meter dash are covered in 2 seconds by a sprinter who starts from rest and accelerates with a constant acceleration. The remaining 90 meters are run with the same velocity the sprinter had after 2 seconds.
    a. Determine the sprinter's constant acceleration during the first 2 seconds.

    b. Determine the sprinter's velocity after 2 seconds have elapsed.

    c. Determine the total time needed to run the full 100 meters.



22. The drawings above represent strobe (flash) photographs of a ball moving in the direction of the arrow. The circles represent the positions of the ball at succeeding instants of time. The time interval between successive positions is the same in all cases.

    a. Rank the magnitude of the ball's average velocity in the last time interval from greatest to least. Explain your reasoning.

    b. Rank the magnitude of the acceleration based on the drawings from greatest to least. Explain your reasoning.

23. Sarah throws a ball straight up in the air with an initial velocity of 19.62 m/s.

    a. How long will the ball be in the air before Sarah catches it? (Assume it is caught at the same height from which it is thrown.)

    b. What maximum height will the ball reach?

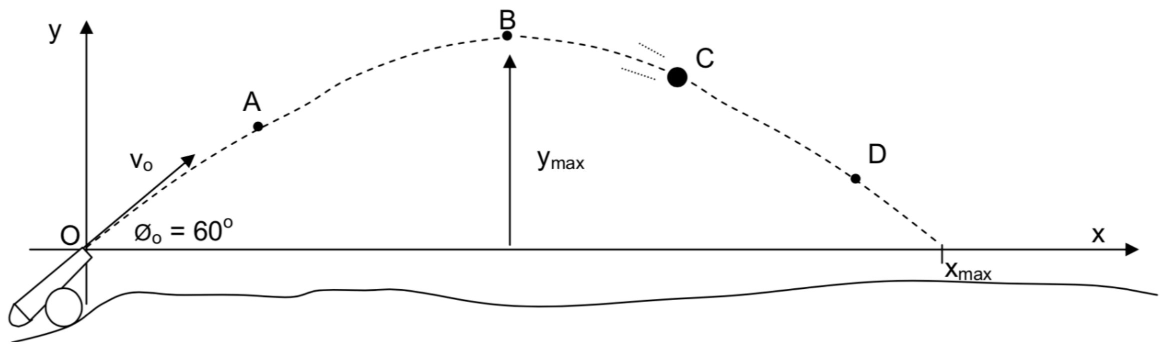
    c. Sketch a position vs. time, velocity vs. time and acceleration vs. time graph representing the ball's motion.

24. A sky diver is using a camera to film his jump. Near the end of his jump, when he is at a height of 50m and falling at a constant rate of 10 m/s, he accidentally drops his camera.

    a. With what velocity does the camera strike the ground?

    b. How many seconds does it take for the camera to strike the ground after it is dropped?

c. What if (for some bizarre reason) the sky diver threw the camera upward with a velocity of 10m/s (relative to the ground) and let the camera fall to the ground. How would the velocity that the camera struck the ground with compare to that of part (a)? How would the time in flight compare to that of part (b)?



25. As part of a Halloween festival, a large pumpkin is fired from a cannon, as shown in the image above. It emerges out of the cannon at an angle of 60 degrees above the horizontal with a speed of 20 m/s. Air resistance is negligible.
    a. At which of the points O, A,B ,C, or D is the magnitude of the vertical component of the pumpkin's velocity (its vertical speed) the greatest? The least?
    b. At which of the points O, A, B, C, or D is the magnitude of the horizontal component of the pumpkin's velocity (its horizontal speed) the greatest? The least?
    c. At which points is the magnitude of the acceleration the greatest? The least?
    d. What is the direction of the pumpkin's acceleration at each point?

26. For each of the following scenarios, sketch position vs. time and a velocity vs. time graphs. Include appropriate numerical scales along both axes. A small amount of computation may be necessary.
    a. Parachutist Jane opens her parachute at an altitude of 1500 meters. She then descends slowly to earth at a steady speed of 15 m/s. Start your graphs as her parachute opens.

b. Trucker Bob starts the day 120 km west of Denver. He drives east for 3 hours at an average 90 km/h before stopping for his coffee break. Let Denver be located at x = 0 km and assume the x-axis points to the east.

c. Sprinter Lisa is in the ready position at the starting line of the 100-meter dash. When the shot is fired, she accelerates from rest at a steady 8 m/s^2 until she crosses the finish line.

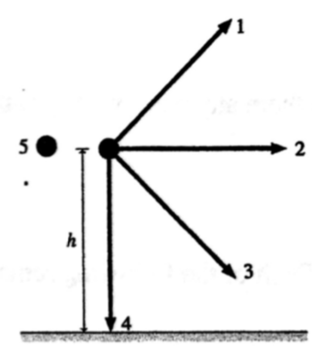
d. Crazy teenage driver Jason is cruising down Research Forest Dr. A stop light turns green and he floors it, accelerating at 10 m/s^2 until he reaches a max speed of 30 m/s. He is only able to maintain this speed for 30 seconds until he encounters another red light and must quickly decelerate to a stop at 20 m/s^2.

27. A cart rolling at a constant velocity fires a ball straight up. Ignore the effects of air resistance.

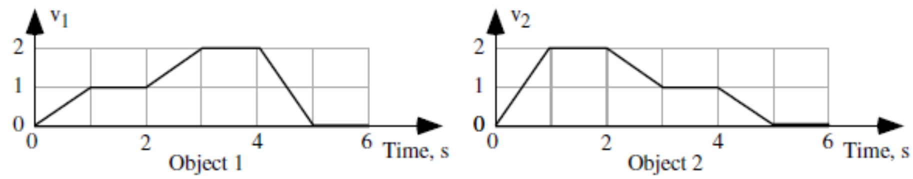
 a. When the ball comes down, will it land in front of the launching tube, behind the launching tube, or directly in it? Explain your answer.

 b. Will your answer change if the cart is accelerating in the forward direction? If so, how?

28. Four balls are simultaneously launched with the same speed from the same height *h* above the ground, as shown to the right. At the same instant, ball 5 is released from rest at the same height. Rank in order, from shortest to longest, the amount of time it takes each of these balls to hit the ground. Explain your answer.



29. The graphs below show the velocity of two objects during the same time interval.



Three students are discussing the displacements of these objects for this interval.

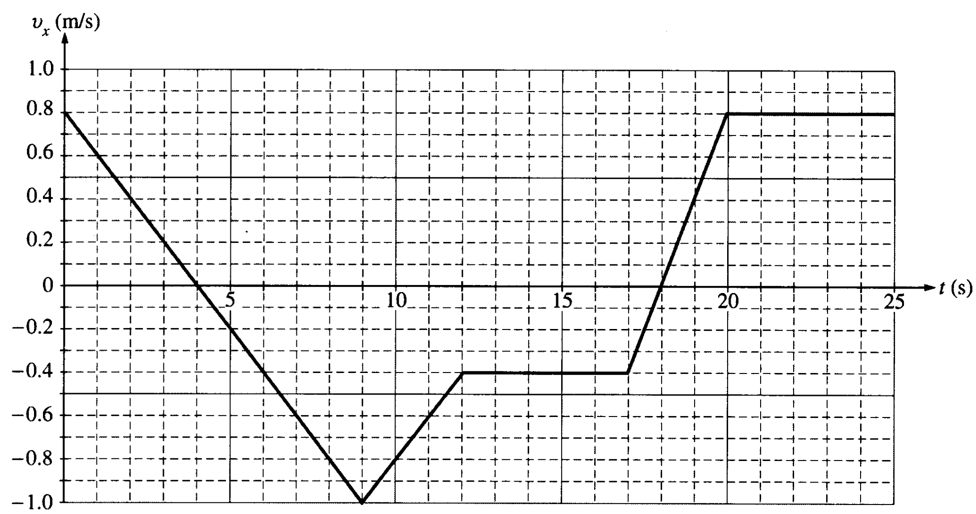
Ariel: "*I think Object 2 will have the greater displacement because it gets to a higher speed faster than Object 1*."

Brody: "*Object 1 spends most of its time speeding up, but object 2 spends most of its time slowing down. Object 1 will go farther*."

Cyrus: "*The displacement is found from the integral or area of the velocity graphs. But in this case we don't know what the integration constant or the initial position is that we need to add to the integral or area. We don't have enough information to find the displacement.*"

Which, if any, of these three students do you agree with? Explain your reasoning.

30. A 0.50 kg cart moves on a straight horizontal track. The graph of velocity vs. time for the cart is given below.



a. Indicate every time t for which the cart is at rest.

b. Indicate every time interval for which the speed (magnitude of velocity) of the cart is increasing.

c. Determine the horizontal position x of the cart at t = 9.0 s if the cart is located at x = 2.0 m at t = 0. d.